

Spectral Theory of Perturbative Decays

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We propose a complex approach to evaluating a function sum of two noncommuting non-Hermitian operators. We propose an explicit expansion of the evolution operator in the case of the neutral K -meson system influenced by an external interaction. The importance is pointed out of considering the algebraic expansion of the time evolution operator whenever the dynamics decouples the internal transitions and center-of-mass motion.

1. INTRODUCTION

The temporal evolution of metastable systems is governed by a non-Hermitian Hamiltonian with nonorthogonal eigenvectors corresponding to complex eigenvalues. The problem of determining an elegant and compact form for the evolution operator

$$\mathcal{U}(t) = \exp[-i\mathcal{H}t], \quad \text{where } \mathcal{H} = \mathcal{H}_0 + \mathcal{V} \quad (1)$$

is connected with the more general issue of expressing explicitly an arbitrary function of the sum of two noncommuting matrix operators. One of the most tantalizing methods for evaluating this matrix function involves the subtleties of complex analysis and has already been developed in the particular case of Hermitian operators (Moretti and Mancini, 1984). The purpose of this paper is to extend this method to non-Hermitian operators. The starting point is the generalized Cauchy formula (Merzbacher, 1968)

$$f(\mathcal{A}) = \frac{1}{2\pi i} \int_{\gamma} f(z)\mathcal{G}(z) dz \quad (2)$$

Here the integral is extended over a contour γ in the complex z plane

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which encloses all eigenvalues of A . It is then possible to obtain an integral expression of $f(A)$ in terms of the well-known resolvent operator

$$\mathcal{G}_A(z) = (z\mathcal{J} - A)^{-1} \quad (3)$$

If A and B are two non-Hermitian and noncommuting operators, the resolvent operators of A and of the sum $A + B$ are given, respectively, by

$$\mathcal{G}_A(z) = (z\mathcal{J} - A)^{-1}, \quad \mathcal{G}(z) = [z\mathcal{J} - (A + B)]^{-1} \quad (4)$$

In the convergence region of the geometric series

$$\left(1 - \frac{B}{z\mathcal{J} - A}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{B}{z\mathcal{J} - A}\right)^n \quad (5)$$

the following expansion for $\mathcal{G}(z)$ holds:

$$\begin{aligned} \mathcal{G}(z) &= \frac{1}{z\mathcal{J} - A} \frac{1}{[z\mathcal{J} - (A + B)]/(z\mathcal{J} - A)} \\ &= \frac{1}{z\mathcal{J} - A} \sum_{n=0}^{\infty} \left(\frac{B}{z\mathcal{J} - A}\right)^n \\ &= \sum_{n=0}^{\infty} \mathcal{G}_A(B^n \mathcal{G}_A) = \sum_{n=0}^{\infty} (\mathcal{G}_A B)^n \mathcal{G}_A \end{aligned} \quad (6)$$

which will be useful later. The right and left eigenvectors of the operator A are defined by the relations

$$A|\Phi_i\rangle = \lambda_i|\Phi_i\rangle, \quad \langle\Psi_i|A = \lambda_i\langle\Psi_i| \quad (7)$$

Contrary to the case where A is Hermitian, the sets $\{|\Phi_i\rangle\}$ and $\{\langle\Psi_i|\}$, although complete, are not orthogonal, and $|\Psi_i\rangle \neq |\Phi_i\rangle$. However, the relation

$$\langle\Psi_i|\Phi_j\rangle = \langle\Psi_i|\Phi_i\rangle \delta_{ij} \quad (8)$$

holds, and therefore it is possible to generalize the completeness relation using the following decomposition of unity:

$$\mathcal{J} = \sum_I \frac{|\Phi_i\rangle\langle\Psi_i|}{\langle\Psi_i|\Phi_i\rangle} \quad (9)$$

We propose to give a spectral expansion of the function $f(A + B)$ in terms of the respective eigenvalues of A and B . If Γ is a closed contour enclosing the whole spectrum of the operator $A + B$, then we have

$$f(A + B) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \mathcal{G}(z) dz \quad (10)$$

and from equation (6)

$$f(A + B) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma} f(z) [(\mathcal{G}_{AB})^n \mathcal{G}_A] dz \quad (11)$$

The matrix elements will be obtained by

$$\begin{aligned} & \langle \Psi_1 | f(A + B) | \Phi_2 \rangle \\ &= \frac{1}{2\pi i} \int_{\Gamma} dz f(z) \sum_{n=0}^{\infty} \langle \Psi_1 | (\mathcal{G}_{AB})^n \mathcal{G}_A | \Phi_2 \rangle \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma} dz f(z) \\ & \quad \times \left\{ \sum_{i_1, i_2, \dots, i_n} \langle \Psi_1 | \mathcal{G}_{AB} \frac{|\Phi_{i_1}\rangle \langle \Psi_{i_1}|}{\langle \Psi_{i_1} | \Phi_{i_1}\rangle} \mathcal{G}_{AB} \frac{|\Phi_{i_2}\rangle \langle \Psi_{i_2}|}{\langle \Psi_{i_2} | \Phi_{i_2}\rangle} \dots \frac{|\Phi_{i_n}\rangle \langle \Psi_{i_n}|}{\langle \Psi_{i_n} | \Phi_{i_n}\rangle} \mathcal{G}_A | \Phi_2 \rangle \right\} \\ &= \sum_{n=0}^{\infty} \sum_{\{i, n\}} \langle \Psi_1 | B \frac{|\Phi_{i_1}\rangle \langle \Psi_{i_1}|}{\langle \Psi_{i_1} | \Phi_{i_1}\rangle} B \frac{|\Phi_{i_2}\rangle \langle \Psi_{i_2}|}{\langle \Psi_{i_2} | \Phi_{i_2}\rangle} \dots \frac{|\Phi_{i_{n-1}}\rangle \langle \Psi_{i_{n-1}}|}{\langle \Psi_{i_{n-1}} | \Phi_{i_{n-1}}\rangle} B | \Phi_2 \rangle \\ & \quad \times \frac{1}{2\pi i} \int_{\Gamma} dz f(z) [(z - \lambda_1)^{-1} (z - \lambda_{i_1})^{-1} \dots (z - \lambda_{i_{n-1}})^{-1} (z - \lambda_2)^{-1}] \end{aligned} \quad (12)$$

where we have used the relation (9) and we have defined $\{i, n\} \equiv \{i_1, i_2, \dots, i_{n-1}\}$. The indices i_k run through the whole set of the eigenvectors as usual, whereas $\langle \Psi_1 |$ and $| \Phi_2 \rangle$ are fixed. If we introduce the function

$$F(z) = f(z) [(z - \lambda_1)(z - \lambda_{i_1}) \dots (z - \lambda_{i_{n-1}})(z - \lambda_2)]^{-1} \quad (13)$$

we can write

$$\begin{aligned} & \langle \Psi_1 | f(A + B) | \Phi_2 \rangle \\ &= \sum_{n=0}^{\infty} \sum_{\{i, n\}} \langle \Psi_1 | B \frac{|\Phi_{i_1}\rangle \langle \Psi_{i_1}|}{\langle \Psi_{i_1} | \Phi_{i_1}\rangle} B \frac{|\Phi_{i_2}\rangle \langle \Psi_{i_2}|}{\langle \Psi_{i_2} | \Phi_{i_2}\rangle} \\ & \quad \times \dots \frac{|\Phi_{i_{n-1}}\rangle \langle \Psi_{i_{n-1}}|}{\langle \Psi_{i_{n-1}} | \Phi_{i_{n-1}}\rangle} B | \Phi_2 \rangle \frac{1}{2\pi i} \int_{\Gamma} dz F(z) \end{aligned} \quad (14)$$

which generalizes the result of Moretti and Mancini (1984). Supposing that the eigenvalues of A are enclosed within Γ , i.e., all the singularities of the function $F(z)$ are inside the integral contour, it is then possible to apply the

theorem of residues. If we denote $\mathcal{R}(\lambda_i)$ as the residue of $F(z)$ at the pole $z = \lambda_i$, the matrix elements in (14) can be rewritten as

$$\begin{aligned} \langle \Psi_1 | f(A + B) | \Phi_2 \rangle &= \sum_{n=0}^{\infty} \sum_{\{i,n\}} \langle \Psi_1 | B \frac{|\Phi_{i_1}\rangle \langle \Psi_{i_1}|}{\langle \Psi_{i_1} | \Phi_{i_1} \rangle} B \frac{|\Phi_{i_2}\rangle \langle \Psi_{i_2}|}{\langle \Psi_{i_2} | \Phi_{i_2} \rangle} \\ &\quad \times \dots \frac{|\Phi_{i_{n-1}}\rangle \langle \Psi_{i_{n-1}}|}{\langle \Psi_{i_{n-1}} | \Phi_{i_{n-1}} \rangle} B | \Phi_2 \rangle \\ &\quad \times \left[\mathcal{R}(\lambda_1) + \sum_{v=1}^{n-1} \mathcal{R}(\lambda_{i_v} + \mathcal{R}(\lambda_2)) \right] \end{aligned} \quad (15)$$

If some eigenvalues are degenerate, the general expression of $F(z)$ is

$$F(z) = f(z) [(z - \lambda_1)^{m_1} (z - \lambda_{i_1})^{m_{i_1}} \dots (z - \lambda_{i_{n-1}})^{m_{i_{n-1}}} (z - \lambda_2)^{m_2}]^{-1} \quad (16)$$

where every exponent m_i is the degeneracy order of the respective eigenvalues λ_i . Finally we want to stress that the set $\{m_i\}$ ($i \in \{1, \dots, n-1\}$) depends on the particular $\{i, n\}$ selected. It is worth noting that these results recover the formulas already present in literature in the limiting case of Hermitian matrices, and they result in a generally more straightforward way than the usual algebraic methods (Friedrichs, 1973; Kato, 1976). The use of these results can be illustrated in the practical example of the evolution operator.

2. THE EVOLUTION OPERATOR OF THE NEUTRAL KAON SYSTEM

The previous results let us take a decisive step toward a complete understanding of the controversial results about the dynamical behavior of a decaying system described by the vector state $|\Psi(t)\rangle$. Its time evolution can be written by means of an operator \mathcal{U} :

$$|\Psi(t)\rangle = \mathcal{U}(t) |\Psi(0)\rangle \quad (17)$$

which can be expressed in the well-known exponential form (using units $\hbar = 1$)

$$\mathcal{U}(t) = \exp[-i\mathcal{H}t] \quad (18)$$

Although the Hamiltonian of a sensible quantum system is expected to be a Hermitian operator, under suitable conditions we may recover the time evolution according to an effective non-Hermitian Hamiltonian as in the case of metastable states. A celebrated example where this description has proved

extremely useful is the two-state kaon complex. If this system is influenced by an external interaction, the Hamiltonian operator can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} \quad (19)$$

where \mathcal{H}_0 and the perturbation \mathcal{V} are, in general, two non-Hermitian and noncommuting operators. Thus, we can apply the formulas of the previous section to expand $\mathcal{U}(t)$ in terms of the eigenfunctions $|K_S\rangle$, $|K_L\rangle$ of \mathcal{H}_0

$$\mathcal{H}_0|K_S\rangle = \lambda_S|K_S\rangle, \quad \mathcal{H}_0|K_L\rangle = \lambda_L|K_L\rangle \quad (20)$$

where $|K_S\rangle$ and $|K_L\rangle$ are the right eigenvectors. The same matrix operator \mathcal{H}_0 also has two left eigenvectors with the same eigenvalues

$$\langle K'_S|\mathcal{H}_0 = \lambda_S\langle K'_S|, \quad \langle K'_L|\mathcal{H}_0 = \lambda_L\langle K'_L| \quad (21)$$

The set $\{|K'_S\rangle, |K'_L\rangle\}$ is the reciprocal set of $\{|K_S\rangle, |K_L\rangle\}$ in the sense that

$$\langle K'_S|K_L\rangle = 0 = \langle K'_L|K_S\rangle \quad (22)$$

If we normalize all the eigenvectors to 1 and denote the overlap as

$$\chi = \langle K_L|K_S\rangle \quad (23)$$

then

$$|K'_S\rangle = \frac{1}{\sqrt{1 - |\chi|^2}} (|K_S\rangle - \chi|K_L\rangle) \quad (24)$$

$$|K'_L\rangle = \frac{1}{\sqrt{1 - |\chi|^2}} (|K_L\rangle - \chi^*|K_S\rangle)$$

where

$$\langle K'_S|K_S\rangle = \sqrt{1 - |\chi|^2} = \langle K'_L|K_L\rangle \quad (25)$$

and

$$\langle K'_L|K'_S\rangle = -\chi \quad (26)$$

Therefore, we have the following equivalent decomposition of unity:

$$\begin{aligned} \mathcal{F} &= |K_S\rangle\langle K_S| + |K'_L\rangle\langle K'_L| \\ &= |K_L\rangle\langle K_L| + |K'_S\rangle\langle K'_S| \\ &= \frac{1}{\sqrt{1 - |\chi|^2}} (|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|) \\ &= \frac{1}{\sqrt{1 - |\chi|^2}} (|K'_S\rangle\langle K_S| + |K'_L\rangle\langle K_L|) \end{aligned} \quad (27)$$

In the particular case of the evolution operator, the generalized Cauchy formula is written in the form

$$\mathcal{U}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-izt} \mathcal{G}(z) dz \quad (28)$$

where the resolvent operator is

$$\mathcal{G}(z) = \frac{1}{z\mathcal{F} - \mathcal{H}} = \frac{1}{z\mathcal{F} - [\mathcal{H}_0 + \mathcal{V}]} \quad (29)$$

and Γ is a closed curve encircling all the complex eigenvalues of the total Hamiltonian \mathcal{H} . If

$$\mathcal{G}_0(z) = \frac{1}{z\mathcal{F} - \mathcal{H}_0} \quad (30)$$

is the resolvent operator of \mathcal{H}_0 , an analogous expansion to equation (6) for $\mathcal{G}(z)$ holds by substituting \mathcal{H}_0 and \mathcal{V} for the operators A and B , respectively. Now, the matrix elements will be obtained by

$$U_{\alpha\beta} = \langle K'_\alpha | \exp[-i\mathcal{H}t] | K_\beta \rangle \quad (31)$$

where the Greek letters α and β are fixed and $\alpha, \beta \in \{S, L\}$. From equation (12) we have

$$\begin{aligned} & \langle K'_\alpha | \exp[-i\mathcal{H}t] | K_\beta \rangle \\ &= \sum_{n=0}^{\infty} \sum_{\{\mu, n\}} \langle K'_\alpha | \mathcal{V} \frac{|K_{\mu_1}\rangle \langle K'_{\mu_1}|}{\langle K'_{\mu_1} | K_{\mu_1} \rangle} \mathcal{V} \frac{|K_{\mu_2}\rangle \langle K'_{\mu_2}|}{\langle K'_{\mu_2} | K_{\mu_2} \rangle} \dots \frac{|K_{\mu_{n-1}}\rangle \langle K'_{\mu_{n-1}}|}{\langle K'_{\mu_{n-1}} | K_{\mu_{n-1}} \rangle} \mathcal{V} | K_\beta \rangle \\ & \quad \times \frac{1}{2\pi i} \int_{\Gamma} dz e^{-izt} [(z - \lambda_\alpha)^{-1} (z - \lambda_{\mu_1})^{-1} \dots (z - \lambda_{\mu_{n-1}})^{-1} (z - \lambda_\beta)^{-1}] \\ &= \sum_{n=0}^{\infty} \sum_{\{\mu, n\}} \langle K'_\alpha | \mathcal{V} \frac{|K_{\mu_1}\rangle \langle K'_{\mu_1}|}{\langle K'_{\mu_1} | K_{\mu_1} \rangle} \mathcal{V} \frac{|K_{\mu_2}\rangle \langle K'_{\mu_2}|}{\langle K'_{\mu_2} | K_{\mu_2} \rangle} \dots \frac{|K_{\mu_{n-1}}\rangle \langle K'_{\mu_{n-1}}|}{\langle K'_{\mu_{n-1}} | K_{\mu_{n-1}} \rangle} \mathcal{V} | K_\beta \rangle \\ & \quad \times \left[\mathcal{R}(\lambda_\alpha) + \sum_{j=1}^{n-1} \mathcal{R}(\lambda_{\mu_j}) + \mathcal{R}(\lambda_\beta) \right] \end{aligned} \quad (32)$$

Here $\{\mu, n\} \equiv \{\mu_1, \mu_2, \dots, \mu_{n-1}\}$ and the indices μ_k vary in the set

$\{S, L\}$. We can give a more explicit expression to the quantity $\langle K'_\alpha | \exp[-i\mathcal{H}t] | K_\beta \rangle$ with the use of formulas (23)–(25)

$$\begin{aligned}
 U_{\alpha\beta} &= \langle K'_\alpha | \exp[-i\mathcal{H}t] | K_\beta \rangle \\
 &= \sum_{n=0}^{\infty} \frac{1}{[\sqrt{1 - |\chi|^2}]^{(n-1)}} \langle K'_\alpha | \mathcal{V} [|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|] \mathcal{V} \\
 &\quad \times [|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|] \dots [|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|] \mathcal{V} |K_\beta \rangle \\
 &\quad \times [\mathcal{R}(\lambda_\alpha) + \sum_{\nu=1}^{n-1} \mathcal{R}(\lambda_{\mu_\nu}) + \mathcal{R}(\lambda_\beta)] \quad (33)
 \end{aligned}$$

Since λ_α , λ_β , and λ_{μ_ν} ($\nu \in \{1, 2, \dots, n-1\}$) are equal to λ_S or λ_L , equation (33) will therefore be rewritten as

$$\begin{aligned}
 U_{\alpha\beta} &= \sum_{n=0}^{\infty} \frac{1}{[\sqrt{1 - |\chi|^2}]^{(n-1)}} \langle K'_\alpha | \mathcal{V} [|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|] \mathcal{V} \\
 &\quad \times [|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|] \dots [|K_S\rangle\langle K'_S| + |K_L\rangle\langle K'_L|] \\
 &\quad \times \mathcal{V} |K_\beta \rangle [\mathcal{R}(\lambda_S) + \mathcal{R}(\lambda_L)] \quad (34)
 \end{aligned}$$

where $\mathcal{R}(\lambda_i)$ is the residue at $z = \lambda_i$, $\lambda_i \in \{\lambda_S, \lambda_L\}$, of the function

$$F(z) = e^{-izt} [(z - \lambda_S)^r (z - \lambda_L)^s]^{-1} \quad (35)$$

and r and s are positive integer numbers subject to the condition $r + s = n + 1$. In a general theory

$$\lambda_{S,L} = \frac{\text{tr } \mathcal{H}_0 \mp \sqrt{[\text{tr } \mathcal{H}_0]^2 - 4 \det \mathcal{H}_0}}{2} \quad (36)$$

so $\lambda_S \neq \lambda_L$ and all eigenvalues are not degenerate. In this way if $n > 2$, λ_S and λ_L are not simple poles for $F(z)$ and a direct calculation of residues gives

$$\mathcal{R}(\lambda_S) = \frac{1}{(r-1)!} \frac{d^{r-1}}{dz^{r-1}} [e^{-izt} (z - \lambda_S)^{-s}] \Big|_{z=\lambda_S} \quad (37)$$

and

$$\mathcal{R}(\lambda_L) = \frac{1}{(s-1)!} \frac{d^{s-1}}{dz^{s-1}} [e^{-izt} (z - \lambda_L)^{-r}] \Big|_{z=\lambda_L} \quad (38)$$

We illustrate the use of these formulas to calculate the second-order term in the expansion in the particular case $\alpha = S$ and $\beta = L$. We have

$$\begin{aligned}
 U_{S,L}^{(2)} &= \frac{1}{\sqrt{1-|\chi|^2}} \left[\langle K_S^{\dagger} | \mathcal{V} | K_S \rangle \langle K_S^{\dagger} | \mathcal{V} | K_L \rangle [\mathcal{R}_{(2)}(\lambda_S) + \mathcal{R}_{(1)}(\lambda_L)] \right. \\
 &\quad \left. + \langle K_S^{\dagger} | \mathcal{V} | K_L \rangle \langle K_L^{\dagger} | \mathcal{V} | K_L \rangle [\mathcal{R}_{(1)}(\lambda_S) + \mathcal{R}_{(2)}(\lambda_L)] \right] \\
 &= \frac{1}{\sqrt{1-|\chi|^2}} \langle K_S^{\dagger} | \mathcal{V} | K_S \rangle \langle K_S^{\dagger} | \mathcal{V} | K_L \rangle \left[e^{-i\lambda_S t} [(\lambda_L - \lambda_S)^{-1} + (\lambda_L - \lambda_S)^{-2}] \right. \\
 &\quad \left. + e^{-i\lambda_L t} (\lambda_L - \lambda_S)^{-2} \right] \\
 &\quad + \frac{1}{\sqrt{1-|\chi|^2}} \langle K_S^{\dagger} | \mathcal{V} | K_L \rangle \langle K_L^{\dagger} | \mathcal{V} | K_L \rangle \left[e^{-i\lambda_L t} [(\lambda_S - \lambda_L)^{-1} + (\lambda_S - \lambda_L)^{-2}] \right. \\
 &\quad \left. + e^{-i\lambda_S t} (\lambda_S - \lambda_L)^{-2} \right] \tag{39}
 \end{aligned}$$

Here we have labeled $\mathcal{R}(\lambda_i)$ with the subscripts (1) and (2) to stress that λ_i is a first- or a second-order pole. Obviously, the presence of higher order terms involves higher order poles, but in the case of the neutral K -meson system, the expression of $U_{\alpha\beta}$ is not so cumbersome as in the general case. In view of this consideration, this perturbative approach is extremely useful in the description of evolution of the two-state kaon complex (Cocolicchio and Viggiano, 1997). But, as mentioned in Moretti and Mancini (1984), the use of the complex analysis is more convenient and it can be successfully applied also when the usual time-dependent perturbation theory fails.

3. CONCLUDING REMARKS

Outside the realm of particle physics, there are many other cases of unstable systems influenced by external interactions where the previous approach becomes indispensable. For example, in modern quantum optics, it seems particularly important to analyze the (para)magnetic resonance (Raby *et al.*, 1954; Feynman *et al.*, 1957) and in general to describe the two states (spin up, spin down) involving electrons and protons with dissipation. It provides the theoretical framework to study a multitude of effects involving laser dynamics. Nevertheless, unstable two-level systems in interaction with other degrees of freedom require the strategy outlined before. The system is

in fact an open system and its dynamical behavior under the influence of an external interaction can be described only by redefining the evolution operator

$$\mathbb{O}(t)\mathcal{U}(t)\mathbb{O}^{-1}(0) = \mathcal{U}(t) = \exp[-i\int\mathcal{H}/\hbar] = e^{A+B} \quad (40)$$

The transformation $\mathbb{O}(t)$ decouples the internal degrees of freedom from the motion of the center of mass and provides a time-independent Hamiltonian $\mathcal{H} = \mathbb{O}\mathcal{H}\mathbb{O}^{-1} - i\hbar\dot{\mathbb{O}}\mathbb{O}^{-1}$. The operators A and B are introduced to clarify the mathematical structure of the calculation below. The new time evolution operator $\mathcal{U}(t)$ is then determined by the exponential factorization of two noncommuting (sometimes non-Hermitian) operators. The importance of the method outlined above is evident now and turns out to be particularly compelling as far as the physical interpretation is concerned. It is worth discussing in connection with the algebraic approach. This method makes use of the parametric differentiation of the exponential of an operator and of the commutation relations in the context of the Baker–Campbell–Hausdorff (BCH) formula (Wilcox, 1967; Lutzky, 1968; Oteo, 1991). Then we can think of separating the center-of-mass A part of the evolution by factorizing $\mathcal{U}(t)$ according to

$$\mathcal{U}(t) = e^A W(t) \quad (41)$$

Thus the complete time evolution of the two-level system is then based on the remaining determination of the operator $W(t)$ which contains the influence of the external interactions on the internal dynamics. At this point, to work out $W(t)$ of equation (41) we consider the operator

$$\mathcal{G}(\lambda) = \exp[\lambda(A + B)] = e^{\lambda A} W(\lambda) \quad (42)$$

and restrict consideration to $\lambda = 1$ at the end. Differentiation of equation (42) with respect to λ leads to the differential equation

$$\frac{dW}{d\lambda} = (e^{-\lambda A} B e^{\lambda A}) W(\lambda) \simeq (B - \lambda[A, B]) W(\lambda) \quad (43)$$

with the initial condition $W(\lambda = 0) = \mathbf{1}$ and using the identity

$$e^{-\lambda A} B e^{\lambda A} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} K_n, \quad K_0 = B, \quad K_{n+1} = [K_n, A] \quad (44)$$

which holds for any two operators A, B . Under general assumptions, it may be written as a matrix equation of the form

$$\begin{pmatrix} dW_{11}/d\lambda & dW_{12}/d\lambda \\ dW_{21}/d\lambda & dW_{22}/d\lambda \end{pmatrix} = \begin{pmatrix} P - \lambda[Q, P] & R \\ R & -P + \lambda[Q, P] \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad (45)$$

This result assumes that the factorization is able to select the nonvanishing commutator $[A, B] = [Q, P]$, which is, however, a c -number, whereas R is the remaining part in B . An example is

$$\begin{aligned} A &= Q\mathbf{1} \\ B &= P\sigma_3 + R\sigma_1 \end{aligned} \quad (46)$$

Equation (45) is an operator-valued system of differential equations, but it contains only commuting operators, so that we can treat it as an ordinary differential equation with the initial condition $W_{ij}(\lambda = 0) = 1$.

Inserting the equation for $dW_{11}/d\lambda$ into the equation for $dW_{21}/d\lambda$ and similarly the one for $dW_{22}/d\lambda$ into that for $dW_{12}/d\lambda$, one gets

$$\begin{aligned} \frac{d^2 W_{11}}{d\lambda^2} &= \{R^2 - [Q, P] + (\lambda[Q, P] - P)^2\} W_{11} \\ \frac{d^2 W_{22}}{d\lambda^2} &= \{R^2 + [Q, P] + (\lambda[Q, P] - P)^2\} W_{22} \end{aligned} \quad (47)$$

After the introduction of the parameter

$$\theta = \frac{R^2}{2[Q, P]} \quad (48)$$

and the change of the variable $y = (\lambda[Q, P] - P) \sqrt{2/[Q, P]}$, equations (47) become

$$\begin{aligned} \frac{d^2 W_{11}}{dy^2} &= \left\{ \frac{y^2}{4} + \theta - \frac{1}{2} \right\} W_{11}(y) \\ \frac{d^2 W_{22}}{dy^2} &= \left\{ \frac{y^2}{4} + \theta + \frac{1}{2} \right\} W_{22}(y) \end{aligned} \quad (49)$$

with the initial conditions $W_{11}(\lambda = 0) = W_{22}(\lambda = 0) = 1$ and

$$\left. \frac{dW_{11}}{dy} \right|_{\lambda=0} = - \left. \frac{dW_{22}}{dy} \right|_{\lambda=0} = \frac{P}{2} \sqrt{\frac{2}{[Q, P]}} \quad (50)$$

The solution of equations (49) is a linear combination of parabolic cylinder functions. The total operator $\mathcal{U}(t)$ is given by

$$\mathcal{U}(t) = \exp \{A\} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad (51)$$

It remains to cancel the initial unitary transformation $\mathbb{O}(t)$ in equation (40) to obtain the exact expression for the time evolution operator:

$$\mathcal{U}(t) = \mathbb{O}^{-1}(t)\mathcal{U}(t)\mathbb{O}(0) = \exp\{A\} \mathbb{O}^{-1}(t)W(t)\mathbb{O}(0) \quad (52)$$

An instructive consistency check is to turn off the external interaction by setting θ to zero. In this case the stable two-level system should be recovered. $\theta = 0$ implies immediately

$$\mathcal{U}(t) = \exp\{-it\mathcal{H}\} \exp\{-itV\} \quad (53)$$

as to be expected. The first term describes the free motion of the system, and the second term contains the internal transitions. It is also possible to derive an expansion for small θ . But the result is then difficult to understand since it contains various combinations of error functions.

In this paper we have analyzed the dynamical evolution of unstable systems under the influence of external interactions. A generalization of the complex spectral theory has been proposed to account for these unstable open systems. The results of the coupled dynamics of the internal transitions and the center-of-mass motion have been worked out with the algebraic expansion of the time evolution operator.

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